

# THE SUBSUM SET OF A NULL SEQUENCE

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ABSTRACT. Given a sequence  $\{x_i\}_{i=1}^{\infty}$  converging to zero, we consider the set  $\Sigma(\{x_i\}_{i=1}^{\infty})$  of numbers which are sums of (infinite, finite, or empty) subsequences of  $\{x_i\}_{i=1}^{\infty}$ . When the original sequence is not absolutely summable,  $\Sigma(\{x_i\}_{i=1}^{\infty})$  is an unbounded closed interval which includes zero. When it is absolutely summable  $\Sigma(\{x_i\}_{i=1}^{\infty})$  is one of the following: a finite union of (nontrivial) compact intervals, a Cantor set, or a “symmetric Cantorval”.

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Recently, while trying to think up some challenging problems for my undergraduate Real Analysis students, I stumbled onto an elementary and, I think, natural question on which I was unaware of any literature.

One of the most counterintuitive facts in the elementary theory of series is that, even if a sequence of real numbers  $\{x_i\}$  converges to zero (that is, it is a **null sequence**), the corresponding *series*  $\sum_{i=1}^{\infty} x_i$  might diverge. The example of this which most of us encounter first is the *harmonic sequence*:

$$\frac{1}{k} \rightarrow 0$$

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for which, surprisingly,

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

However, if we throw away enough of these terms—for example, if we throw away all reciprocals of numbers which are not powers of two—we end up with a sequence whose corresponding series *does* converge. We will call such a sequence a *summable subsequence* of our original sequence, and its sum a **subsum**, of our (original) sequence. Then we might ask about the set of all possible subsums of a given sequence (assuming always that the original sequence goes to zero): is it an interval, a finite union of intervals... or something more complicated?

This turns out to be a challenging question: I set out trying to answer it and came to a number of interesting conclusions, but was unable to give a satisfactory general description of such sets on my own. However, a bibliographic suggestion by Michał Misiurewicz, in response to an earlier version of the present paper, led me by chance to a 1988 paper by J. A. Guthrie and J. E. Nymann [8], which gives a complete topological description of subsum sets as well as a review of some earlier work on the problem.<sup>1</sup> This involves an interesting interplay between standard topics on sequences and series, some elementary number theory, and the topology of subsets of the line, which provides an appealing “extra topic” for undergraduate analysis students.

Most of our discussion will focus on *positive* null sequences, which can be studied using geometric ideas. However, we shall see toward the end of this paper how the description of *all* subsum sets can be reduced to the corresponding description of subsum sets for positive null sequences.

## 1. POSITIVE, NON-SUMMABLE SEQUENCES

We can think of the harmonic sequence as an infinite collection of dominoes of successively shorter lengths: the  $k^{th}$  domino has length  $\frac{1}{k}$ . The fact that the series diverges means that if we put them all end-to-end, we will fill out a whole half-line.

Now suppose we are given a positive real number  $r$ . Can we find a collection of dominoes from this set which exactly fill up an interval of length  $r$ ?

Well, we know the lengths of the dominoes are going to zero, so except for the first few, they are all shorter than  $r$ . This means that (excluding the first few) we can fit any one of them inside the interval. But actually, if we exclude a few more, they are all less than *half*  $r$  in length. This means we

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<sup>1</sup>I was gratified to discover that the terminology I had adopted in my musings on the subject is almost identical to that used in most earlier writing on the subject. The one substantial exception is the word “Cantorval”, coined by two Brazilians in [13], which evokes for me the Samba on Fat Tuesday in Rio...

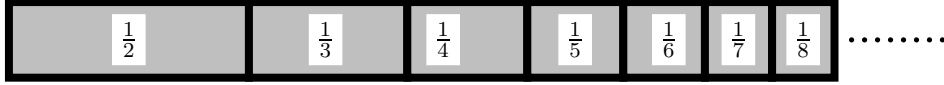
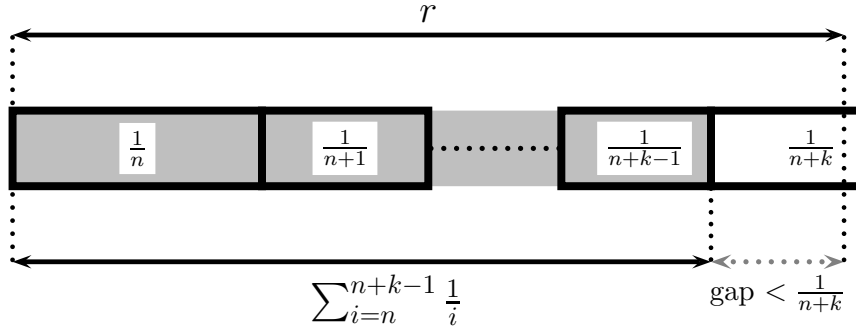


FIGURE 1. Harmonic Dominoes

can start anywhere (beyond our excluded ones) and fit at least *two successive dominoes* inside the interval. Extrapolating, with the exception of some (but finitely many) of the first ones, we can, by starting far enough down the line, fit a string of *any specified finite number of successive dominoes* inside the interval. If we start with the  $n^{\text{th}}$  domino and fit in as many successive dominoes as we can (starting from the  $n^{\text{th}}$ ), then the first domino that “pokes out” will certainly be shorter than  $\frac{1}{n}$ . In fact, if we have managed to squeeze in  $k$  dominoes (starting from the  $n^{\text{th}}$ ) but cannot fit the next one in, then the one that pokes out has length  $\frac{1}{n+k}$ . This means that the ones we *can* fit fill an interval that is *shorter* than  $r$ —but its length *plus*  $\frac{1}{n+k}$  is *more* than  $r$ . It follows that after we have squeezed in  $k$  successive dominoes starting from the  $n^{\text{th}}$ , we are left with an unfilled gap which is shorter than  $\frac{1}{n+k}$ . (Figure 2)

FIGURE 2. Trying to fill up  $r$ 

Now, we look for more dominoes, to fill this gap. We start further down our list of dominoes, finding a set of  $k'$  successive ones, starting with the  $(n')^{\text{th}}$  (where  $n' \gg n+k$ ), that fill out our gap—except for a new, smaller gap of length less than  $\frac{1}{n'+k'}$ . And we continue. With a little more care, we can choose our starting point at each stage so that the size of the gap is cut to less than half its current value with each new filling. When we are all done, we have created a subsequence of our dominoes whose combined total length is *exactly*  $r$ .

Let's look back at what we did. We didn't really use any special properties of the harmonic sequence in this construction, other than the fact that the

lengths of the dominoes go to zero, but their sum diverges (to infinity). So we have a theorem:

**Theorem 1.** *If  $\{x_i\}$  is a positive null sequence for which  $\sum_{i=1}^{\infty} x_i = \infty$ , then every  $r > 0$  is the sum of some subsequence of  $\{x_i\}$ .*

Actually, there is one minor technical point we need to note here. When thinking about the harmonic sequence, we *did* take advantage of the fact that it is decreasing. In general, the sequence we are looking at might be presented in an order which is not decreasing. Fortunately, for a sequence of *positive* numbers, the sum (of the series) is not changed by rearranging their order. (This was noted by Dirichlet in 1837 [5, p. 315] without explicit proof; a proof can be found in many basic analysis books, for example [18, Thm. 3.56, p. 68].<sup>2</sup>). Intuitively, the total length of a collection of dominoes set end-to-end is not changed if we set them down in a different order. This means we can work with them in a *non-increasing* order:  $x_{k+1} \leq x_k$  for every  $k$ . This will be an implicit assumption in all of our reasoning, at least while looking at positive sequences:

**Standing Assumption:** *When dealing with positive sequences, we assume (without loss of generality) that the given sequence is non-increasing:*

$$x_{i+1} \leq x_i \text{ for all } i = 1, 2, \dots$$

## 2. POSITIVE SUMMABLE SEQUENCES

OK, so we have answered our question for a sequence of *positive* numbers going to zero whose sum *diverges*. What about if the sum *converges*?

We start with two examples.

First, consider the sequence of (positive integer) powers of  $\frac{1}{2}$

$$x_i = \frac{1}{2^i}, \quad i = 1, 2, \dots$$

which sums to

$$\sum_{i=1}^{\infty} x_i = \frac{1/2}{1 - 1/2} = 1.$$

We can again picture our sequence as a collection of dominoes (the  $i^{th}$  has length  $(\frac{1}{2})^i$ ); clearly, since *all* of them placed end-to-end fill an interval of length 1, any subcollection will fill a shorter interval; that is, any subsum belongs to the interval  $[0, 1]$ . Now, expressing a number in  $[0, 1]$  as a sum of (distinct) powers of  $\frac{1}{2}$  is the same as giving its binary or *base 2* expansion: to be more precise, a binary sequence (sequence of zeroes and ones)

$$\xi = \{\xi_i\}_{i=1}^{\infty}$$

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<sup>2</sup>The basic idea is that the partial sums for any ordering are themselves a strictly increasing sequence, and any particular partial sum for *one* ordering can be bracketed between two partial sums of any *other* particular order, so the two limits are the same.

(each  $\xi_i$  is 0 or 1) corresponds to the number

$$x(\xi) = \sum_{i=1}^{\infty} \frac{\xi_i}{2^i}.$$

Every number between 0 and 1 has a binary expansion, so the subsum set in this case<sup>3</sup> is an interval with endpoints 0 and 1.

Now, consider the sequence of powers of  $\frac{1}{3}$

$$x_i = \frac{1}{3^i}, \quad i = 1, 2, \dots$$

which sums to

$$\sum_{i=1}^{\infty} x_i = \frac{1/3}{1 - 1/3} = \frac{1}{2}.$$

As before, any subsum belongs to the interval  $[0, \frac{1}{2}]$ . But on closer inspection, it becomes clear that not every point in this interval occurs as a subsum. For example, any subsum which does *not* involve the first term,  $\frac{1}{3}$ , is at most equal to

$$X_1 := \sum_{i>1} \frac{1}{3^i} = \frac{1/9}{1 - 1/3} = \frac{1}{6}$$

and hence belongs to the interval

$$J_0 := \left[0, \frac{1}{6}\right]$$

whereas any subsum which *does* involve the first term belongs to

$$J_1 := \left[\frac{1}{3}, \frac{1}{2}\right].$$

Note that  $J_1$  is the translate of  $J_0$  by  $x_1 = \frac{1}{3}$ , and the set of subsums is actually contained in the union of two disjoint closed intervals

$$\mathfrak{C}_1 := J_0 \cup J_1.$$

That is, distinguishing subsums according to whether they do or don't involve the first term of the sequence breaks the set of all subsums into two pieces, the second a translate of the first.

Now we can continue this analysis by looking as well at whether or not the *second* term of our sequence,  $\frac{1}{9}$ , is involved. A subsum which does not involve either  $\frac{1}{3}$  or  $\frac{1}{9}$  is at most

$$X_2 := \sum_{i>2} \frac{1}{3^i} = \frac{1/27}{1 - 1/3} = \frac{1}{18}$$

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<sup>3</sup>We shall see later that this needs some clarification: see Definition 2.

and so belongs to the closed interval

$$J_{00} := \left[0, \frac{1}{18}\right]$$

while one which involves  $\frac{1}{9}$  but *not*  $\frac{1}{3}$  belongs to

$$J_{01} := \frac{1}{9} + J_0 = \left[\frac{1}{9}, \frac{1}{6}\right];$$

that is, the subsums which do not involve the first term of the sequence break into two parts—they are all contained in the union of two disjoint closed subintervals of  $J_0$

$$J_{00} \cup J_{01} \subset J_0.$$

Similarly, the subsums which *do* involve the first term split into those which *don't* involve the second, all of which belong to the closed interval

$$J_{10} := \frac{1}{3} + J_{00} = \left[\frac{1}{3}, \frac{7}{18}\right]$$

and those that *do* involve *both* of the first two terms, which belong to

$$J_{11} := \frac{1}{3} + \frac{1}{9} = \left[\frac{4}{9}, \frac{1}{2}\right].$$

Thus, when we take account of all the possibilities for which of the first two terms of the sequence occur in a given subsum, we find that the set of subsums is contained in the union of four disjoint closed intervals (Figure 3)—two subintervals of  $J_0$  and two subintervals of  $J_1$ :

$$\begin{aligned} \mathfrak{C}_2 &:= J_{00} \cup J_{01} \cup J_{10} \cup J_{11} \\ &= \left[0, \frac{1}{18}\right] \cup \left[\frac{1}{9}, \frac{1}{6}\right] \cup \left[\frac{1}{3}, \frac{7}{18}\right] \cup \left[\frac{4}{9}, \frac{1}{2}\right]. \end{aligned}$$

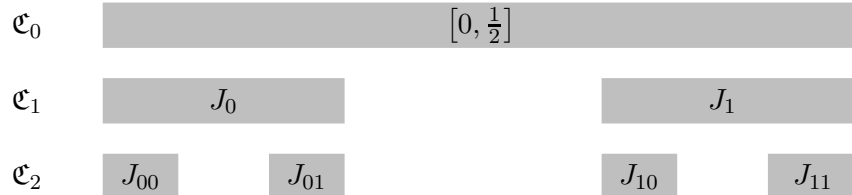


FIGURE 3. The sets  $\mathfrak{C}_n$

Of course, we can continue this process. A subsequence of  $\{x_i\}$  can be specified using the sequence  $\xi = \xi_1\xi_2\cdots$  of zeroes and ones defined by

$$(1) \quad \xi_k = \begin{cases} 1 & \text{if } x_k \text{ is included in the subsequence,} \\ 0 & \text{if it is not.} \end{cases}$$

The sum corresponding to this subsequence is then

$$(2) \quad x_\xi := \sum_{k=1}^{\infty} \xi_k \cdot x_k.$$

For our particular example, this reads

$$x_\xi = \sum_{k=1}^{\infty} \frac{\xi_k}{3^k}$$

which is a base three expansion for  $x_\xi$ .

The intervals  $J_{00}, J_{01}, J_{10}$  and  $J_{11}$  result from sorting the subsum set according to which of the first two terms of the sequence  $\{x_i\} = \{x_i\}_{i=1}^{\infty}$  are included in a given subsum—that is, according to the initial “word” of length 2 in the defining sequence  $\xi$ . In general, we can parse any subsum into an initial finite sum,  $x_{\xi_1\cdots\xi_n}$  determined by the initial “word”  $\xi_1\cdots\xi_n$  of length  $n$ , and the rest of the sum, which is a subsum of the sequence  $\{x_i\}_{i=n+1}^{\infty}$  obtained by omitting the first  $n$  terms of  $\{x_i\}_{i=1}^{\infty}$ . Let us informally<sup>4</sup> denote the subsum set of a sequence  $\{x_i\}_{i=1}^{\infty}$  by  $\Sigma(\{x_i\}_{i=1}^{\infty})$ , and write

$$\Sigma_{(n)} := \Sigma(\{x_i\}_{i=n+1}^{\infty})$$

for the set of subsums which do not involve the first  $n$  terms  $x_1, \dots, x_n$ . Then the collection of all subsums whose defining sequence  $\xi$  has initial word  $\xi_1\cdots\xi_n$  can be written

$$x_{\xi_1\cdots\xi_n} + \Sigma_{(n)};$$

letting the initial word of length  $n$  range over all the possible  $n$ -tuples of zeroes and ones, we fill up our subsum set:

$$(3) \quad \Sigma(\{x_i\}_{i=1}^{\infty}) = \bigcup_{\xi_1\cdots\xi_n \in \{0,1\}^n} (x_{\xi_1\cdots\xi_n} + \Sigma_{(n)}).$$

As before,  $\Sigma_{(n)}$  is contained in the closed interval<sup>5</sup>

$$J_{0^n} = [0, X_n]$$

where  $X_n$  is the highest sum in  $\Sigma_{(n)}$

$$X_n = \sum_{k>n} x_k$$

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<sup>4</sup>A formal definition will be given shortly in Definition 2.

<sup>5</sup> $0^n$  denotes the word of length  $n$  consisting of all zeroes.

and it follows that (for each fixed  $n$ ) our whole subsum set is contained in the union of  $2^n$  closed intervals

$$\mathfrak{C}_n = \bigcup_{\xi_1 \dots \xi_n \in \{0,1\}^n} J_{\xi_1 \dots \xi_n}$$

where

$$\begin{aligned} J_{\xi_1 \dots \xi_n} &:= x_{\xi_1 \dots \xi_n} + J_{0^n} \\ &= [x_{\xi_1 \dots \xi_n} + 0, x_{\xi_1 \dots \xi_n} + X_n]. \end{aligned}$$

In our example,

$$X_n = \sum_{k=n+1}^{\infty} \frac{1}{3^k} = \frac{1/3^{n+1}}{1 - \frac{1}{3}} = \frac{1}{2 \cdot 3^n}$$

so

$$J_{0^n} = \left[0, \frac{1}{2 \cdot 3^n}\right].$$

Having fixed an initial word of length  $n$ , we have two possibilities for the next,  $(n+1)^{st}$  entry in  $\xi$ : either  $\xi_{n+1} = 0$  or  $\xi_{n+1} = 1$ . This means that each interval  $J_{\xi_1 \dots \xi_n}$  of  $\mathfrak{C}_n$  contains two subintervals associated to initial words of length  $n+1$  in  $\xi$ :

$$J_{\xi_1 \dots \xi_n 0} = x_{\xi_1 \dots \xi_n} + [0, X_{n+1}]$$

and

$$J_{\xi_1 \dots \xi_n 1} = x_{\xi_1 \dots \xi_n} + \frac{1}{3^{n+1}} + [0, X_{n+1}]$$

where

$$X_{n+1} = \frac{1}{2 \cdot 3^{n+1}}.$$

The important thing to notice is that these two subintervals have the same length,  $X_{n+1}$ , and the second is a translate of the first by an amount greater than  $X_{n+1}$ . This means they are disjoint. Looking a bit more closely, we note that the first subinterval *starts* at the *left* endpoint of  $J_{\xi_1 \dots \xi_n}$  while the second *ends* at its *right* endpoint. Thus, passing from the union  $\mathfrak{C}_n$  of intervals determined by words of length  $n$  to the union  $\mathfrak{C}_{n+1}$  of those determined by words of length  $n+1$ , each component interval of  $\mathfrak{C}_n$  acquires a gap in its middle, separating two subintervals which are components of  $\mathfrak{C}_{n+1}$ . In fact, since  $X_{n+1} = \frac{1}{3}X_n$ , this gap is precisely the “middle third” of each component. Hence we are carrying out the construction of the middle-third Cantor set, except that we start from the interval  $[0, \frac{1}{2}]$  instead of  $[0, 1]$ . In this way, when we pass to the intersection

$$\mathfrak{C}_\infty = \bigcap_{n=1}^{\infty} \mathfrak{C}_n$$



we obtain a version of the standard Cantor set, but scaled down by a factor of a half.

We can also understand this phenomenon analytically. Our subsum set consists of numbers whose base 3 expansion can be written using only summands of the form  $\frac{1}{3^k}$ ; but in general, a base three expansion for a real number may also include summands of the form  $\frac{2}{3^k}$ ; so our subsum set excludes any number which cannot be written in base 3 without using the “digit” 2. Recall that the standard middle-third Cantor set is the set of numbers in  $[0, 1]$  whose base three expansion uses only zeroes and *twos*, whereas our subsum set consists of numbers in  $[0, \frac{1}{2}]$  whose base three expansion involves only zeroes and *ones*. In particular, multiplication of any subsum by 2 changes all the 1’s in its expansion to 2’s, and thus gives an explicit mapping of our subsum set to the standard Cantor set.

The argument above shows that the subsum set of the sequence of powers of  $\frac{1}{3}$  is a Cantor set. However, the construction of the sets  $\mathfrak{C}_n$  and  $\mathfrak{C}_\infty$  applies to *any* positive summable null sequence, with the proviso that in general, the intervals  $J_{\xi_1 \dots \xi_n}$  need not be disjoint—so our final set  $\mathfrak{C}_\infty$  need not be a Cantor set. In fact, for the powers of  $\frac{1}{2}$ , we have  $X_n = \frac{1}{2^n}$ , and the intervals  $J_{\xi_1 \dots \xi_n}$  abut, so  $\mathfrak{C}_n = [0, 1]$  for all  $n$  (and hence for “ $n = \infty$ ”). As we shall see, even more complicated behavior is possible which mixes overlap and disjointness.

In general, though, the procedure we have outlined produces the compact set  $\mathfrak{C}_\infty$ , which is guaranteed to contain our subsum set. But certainly at each finite stage, the set  $\mathfrak{C}_n$  contains more than  $\Sigma(\{x_i\}_{i=1}^\infty)$ . So, what about the intersection?—does  $\Sigma(\{x_i\}_{i=1}^\infty)$  equal  $\mathfrak{C}_\infty$ , or is it a proper subset?

The answer to this hinges on what we mean by a “subsequence”. Usually a “subsequence” of an infinite sequence is understood to itself be infinite. If we use this notion in our definition of subsums, we exclude any number given as a *finite* sum of powers of  $\frac{1}{3}$ —that is, we exclude the left endpoint of each of our intervals  $J_{\xi_1 \dots \xi_k}$ . The resulting set is a bit awkward to describe. So we follow a convention going back to S. Kakeya (whose 1914 paper [10] is the first one I am aware of on this topic) and include *finite* subsequences, as well as the *empty* sequence (whose sum we take to be zero), in our formal definition of the subsum set.

**Definition 2.** *The **subsum set** of a null sequence*

$$x_1, x_2, \dots \rightarrow 0$$

*is the collection*

$$\Sigma(\{x_i\}_{i=1}^\infty)$$

*of all numbers of the form*

$$x_\xi := \sum_{k=1}^{\infty} \xi_k \cdot x_k,$$

where

$$\xi = \xi_1 \xi_2 \cdots$$

is any sequence of zeroes and ones for which the subsequence  $\{\xi_i \cdot x_i\}_{i=1}^\infty$  is summable.<sup>6</sup>

This definition simply codifies the idea that we take sums of infinite, finite, and empty subsequences of  $\{x_i\}_{i=1}^\infty$ . Note that a *finite* subsum corresponds to a sequence  $\xi$  which is eventually all zeroes; we shall often omit the “tail of zeroes” when specifying the sequence  $\xi$  in such a situation.

Now, we have constructed a nested sequence of compact sets  $\mathfrak{C}_n$ , each containing our subsum set; it follows that  $\Sigma(\{x_i\}_{i=1}^\infty)$  is contained in the compact set  $\mathfrak{C}_\infty$ . Furthermore,  $\mathfrak{C}_n$  consists of intervals of length  $X_n$ , each having nonempty intersection with  $\Sigma(\{x_i\}_{i=1}^\infty)$  (for example its endpoints), which means that all points of  $\mathfrak{C}_n$  are within distance  $X_n$  of the set  $\Sigma(\{x_i\}_{i=1}^\infty)$ . Since we have assumed our sequence is summable, its “tails”  $X_n$  must converge to zero, which implies that  $\mathfrak{C}_\infty$  is the *closure* of  $\Sigma(\{x_i\}_{i=1}^\infty)$ .

The construction of  $\mathfrak{C}_\infty$  automatically implies several properties:

- Since  $X_n > 0$ , every component of  $\mathfrak{C}_n$  is an interval, and hence it has no isolated points—it is *perfect*. This property persists under nested intersection, so  $\mathfrak{C}_\infty$  is a perfect set.
- $\mathfrak{C}_n$  is a union of closed intervals  $J_{\xi_1 \dots \xi_n}$  of length  $X_n$ ; in particular, each point of  $\mathfrak{C}_n$  is within distance  $X_n$  of at least one *right* endpoint and at least one *left* endpoint of some  $J_{\xi_1 \dots \xi_n}$ . Since  $X_n \rightarrow 0$ , this means the right (*resp.* left) endpoints of the various intervals  $J_{\xi_1 \dots \xi_n}$  are dense in  $\mathfrak{C}_\infty$ .

In the case of powers of  $\frac{1}{3}$ , we have an explicit homeomorphism between the subsum set  $\Sigma\left(\left\{\frac{1}{3^k}\right\}_{k=1}^\infty\right)$  and the middle-third Cantor set, telling us that this subsum set is compact, and hence *equals*  $\mathfrak{C}_\infty$ . To go beyond this example, we need to show that  $\Sigma(\{x_i\}_{i=1}^\infty)$  is closed in general. This was done in [10] by a direct argument, but we can finesse the general case using the example and a sneaky trick.

For our example (powers of  $\frac{1}{3}$ ), the sequence  $\xi$  for a particular subsum is a base 3 expansion of that subsum, so points of the Cantor set are in one-to-one correspondence with sequences  $\xi$  of zeroes and ones. Furthermore, this mapping is a homeomorphism (points with expansions that agree for a long time are close to each other, and vice-versa). But half of this also applies to a general subsum set: for *any* sequence  $\{x_i\}$ , two subsums whose defining sequences  $\xi$  agree for at least  $n$  places belong to the same interval  $J_{\xi_1 \dots \xi_n}$ , which is an interval of length  $X_n$ . And that length, which is by definition a “tail” of a convergent series, goes to zero. Thus, the mapping taking a point of the (middle-third) Cantor set to its defining sequence  $\xi$  and then to the point  $x_\xi$  in our subsum set corresponding to the same sequence is

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<sup>6</sup>In the context of this section, where we have assumed the original sequence is positive and summable, every subsequence is summable.

continuous. Since it is also onto, we have exhibited a general subsum set  $\Sigma(\{x_i\}_{i=1}^\infty)$  as a continuous image of a compact set—hence it is also compact. From this we can conclude that

$$\mathfrak{C}_\infty = \Sigma(\{x_i\}_{i=1}^\infty).$$

Hence  $\Sigma(\{x_i\}_{i=1}^\infty)$  has the properties noted above for  $\mathfrak{C}_\infty$ : it is a perfect set, and (since the left (*resp.* right) endpoint of any  $J_{\xi_1 \dots \xi_n}$  is the sum of a finite (*resp.* infinite) subsequence), both kinds of sums are dense in  $\Sigma(\{x_i\}_{i=1}^\infty)$ .

$\Sigma(\{x_i\}_{i=1}^\infty)$  also has some symmetry properties. We have already seen that (fixing  $n$ )  $\Sigma(\{x_i\}_{i=1}^\infty)$  is a union of sets  $J_{\xi_1 \dots \xi_n}$  which are just translates of each other; this means that for each fixed  $n$  the sets  $\Sigma(\{x_i\}_{i=1}^\infty) \cap J_{\xi_1 \dots \xi_n}$  are homeomorphic. Another symmetry is the reflection about the midpoint, given by

$$(4) \quad x \mapsto X_0 - x.$$

To see this particular symmetry, note that when  $x$  is a subsum of our sequence defined by the sequence  $\xi$  of 0's and 1's, then  $X_0 - x$  is defined by the sequence  $\tilde{\xi}_i$ , where  $\tilde{\xi}_i = 1 - \xi_i$ —that is,  $X_0 - x$  is the sum of all the terms *not* included in the sum defining  $x$ .

We summarize<sup>7</sup> these general observations in the following theorem:

**Theorem 3.** *For every summable, positive null sequence  $x_1, x_2, \dots \rightarrow 0$  with sum*

$$\sum_{k=0}^{\infty} x_k = X_0,$$

*the subsum set  $\Sigma(\{x_i\}_{i=1}^\infty)$  is a perfect set with convex hull  $[0, X_0]$  which is symmetric under the reflection  $x \mapsto X_0 - x$ .*

*Furthermore, the collection of all sums of finite subsequences (as well as the collection of all sums of infinite subsequences) is dense in  $\Sigma(\{x_i\}_{i=1}^\infty)$ .*

The fact that  $\Sigma(\{x_i\}_{i=1}^\infty)$  is perfect was proved by Shoichi Takeya in 1914 [10] and independently by Hans Hahn in 1941 [9]<sup>8</sup> The reflection symmetry of subsum sets was noted by Hahn, as well as by Joseph Nymann and Ricardo Saenz in [15].

### 3. TERMS VS. TAILS: SUBSUM SETS OF GEOMETRIC AND $p$ -SERIES

In the examples studied so far, we have observed two extremes of behavior. For the powers of  $\frac{1}{3}$ , the intervals  $J_{\xi_1 \dots \xi_n}$  for any fixed  $n$  are disjoint, and in the limit we obtain a Cantor set as  $\mathfrak{C}_\infty$ . But for powers of  $\frac{1}{2}$ , these intervals touch, as a result of which all the sets  $\mathfrak{C}_n$  are the same, and  $\mathfrak{C}_\infty$  is an interval. To understand the basis of these phenomena in general, we examine the recursive step in the construction of  $\mathfrak{C}_\infty$ .

<sup>7</sup>No pun intended.

<sup>8</sup>A 1948 paper by P. Kesava Menon [14] addresses similar issues, but I find it confusing to determine just what is being proved.

When we go from  $\mathfrak{C}_{n-1}$  to  $\mathfrak{C}_n$ , each interval  $J_\xi$  (for a fixed  $(n-1)$ -word  $\xi$ ) is replaced by the union of two subintervals, corresponding to the  $n$ -words  $\xi^- = \xi 0$  and  $\xi^+ = \xi 1$  obtained by appending 0 (*resp.* 1) to  $\xi$ . Both of these subintervals have length equal to the  $n^{\text{th}}$  tail  $X_n$ , and the second is the translate of the first by the  $n^{\text{th}}$  term  $x_n$ . Furthermore, the right (*resp.* left) endpoint of  $J_\xi$  is the same as the right (*resp.* left) endpoint of  $J_{\xi^-}$  (*resp.*  $J_{\xi^+}$ ). Thus we can distinguish two cases:

**Term exceeds Tail:** If  $x_n > X_n$ , the two intervals are disjoint, so  $J_\xi$  in  $\mathfrak{C}_n$  is replaced by a *disjoint* union of subintervals in  $\mathfrak{C}_{n+1}$ ; that is,  $J_\xi$  breaks into the *disjoint* union of  $J_{\xi^-}$  and  $J_{\xi^+}$ , leaving a “gap” in the middle.

**Tail bounds Term:** If  $x_n \leq X_n$ , the two intervals share at least one point, so their union equals  $J_\xi$ .

Note that with  $n$  fixed, the intervals  $J_\xi$  obey the same rule for every  $n$ -word. Also remember that they all have length  $X_n$ , which goes to zero (since  $X_n$  is the tail of a convergent series). This shows

**Theorem 4.** *Suppose  $\{x_i\}$  is a summable sequence of positive real numbers. Then*

- (1) *If  $x_n > X_n$  (i.e., the term exceeds the tail) for every  $n$ , then for each  $n$ ,  $\mathfrak{C}_n$  is the disjoint union of the  $2^n$  closed intervals  $J_\xi$  as  $\xi$  ranges over the words of length  $n$ . It follows that  $\mathfrak{C}_\infty = \Sigma(\{x_i\}_{i=1}^\infty)$  is a Cantor set.*
- (2) *If  $x_n \leq X_n$  (i.e., the tail bounds the term) for every  $n$ , then for each  $n$ ,  $\mathfrak{C}_n = \mathfrak{C}_0 = [0, X_0]$ , so  $\mathfrak{C}_\infty = \Sigma(\{x_i\}_{i=1}^\infty)$  is the interval  $[0, X_0]$ .*

These properties were established by Hornich [9]. Takeya [10] noted the second property (in fact that the tail always bounds the term if and only if the subsum set is an interval—cf. our Lemma 8 and Proposition 9).

For a geometric sequence with first term  $a$  and ratio<sup>9</sup>  $\rho \in (0, 1)$ , we know that

$$x_n = a\rho^{n-1}$$

and

$$X_n = \frac{a\rho^n}{1-\rho}$$

so

$$\frac{X_n}{x_n} = \frac{\rho}{1-\rho}$$

which is at least 1 for  $\rho \geq \frac{1}{2}$  and strictly less than 1 for  $0 < \rho < \frac{1}{2}$ . This immediately gives us a description of  $\Sigma(\{x_i\}_{i=1}^\infty)$  for any positive geometric sequence.

---

<sup>9</sup>(that is, a geometric sequence whose terms are positive and tend to zero)

**Corollary 5.** *If  $\{x_i = a\rho^{i-1}\}$  is a geometric sequence with initial term  $a > 0$  and ratio  $\rho \in (0, 1)$ , then  $\Sigma(\{x_i\}_{i=1}^{\infty})$  is*

- (1) *a Cantor set for  $0 < \rho < \frac{1}{2}$*
- (2) *the interval  $\left[0, X_0 = \frac{a}{1-\rho}\right]$  for  $\frac{1}{2} \leq \rho < 1$ .*

Theorem 4 tells us what happens when only one of the two possible relations between the terms and the tails occurs. What about if both occur, but one of them occurs *eventually*?<sup>10</sup>

As an example, consider the sequence starting with 2 and then followed by the powers of  $\frac{1}{2}$ . We already know that the sequence starting from the second term (*i.e.*, just the powers of  $\frac{1}{2}$ ) has subsum set  $[0, 1]$ , and it follows from Equation (3) that the full subsum set is

$$\mathfrak{C}_{\infty} = [0, 1] \cup (2 + [0, 1]) = [0, 1] \cup [2, 3].$$

These two intervals are disjoint because the first term,  $x_1 = 2$ , is greater than the first tail,  $X_1 = 1$ .

In general, if the tail bounds the term after the  $N^{\text{th}}$  place

$$x_k \leq X_k \text{ for } k > N$$

then Theorem 2 applied to the sequence starting after position  $N$  tells us that

$$\Sigma_{(N)} = [0, X_N]$$

and then Equation (3) tells us that  $\Sigma(\{x_i\}_{i=1}^{\infty})$  is the union of  $2^N$  closed intervals, which means, allowing for some overlaps between them, that it is the *disjoint* union of *at most*  $2^N$  intervals. Furthermore, if the term exceeds the tail for *all* of the first  $K$  places

$$x_k > X_k \text{ for } k = 1, 2, \dots, K$$

then the intervals  $J_{\xi_1 \dots \xi_n}$  are all disjoint, so  $\mathfrak{C}_K$  consists of  $2^K$  disjoint intervals. So in this case  $\mathfrak{C}_{\infty} = \mathfrak{C}_N$  has *at least*  $2^K$  components. Summarizing, we have

**Proposition 6.** *Suppose  $\{x_i\}$  is a positive, summable null sequence.*

- (1) *If the tail bounds the term eventually, then  $\mathfrak{C}_{\infty}$  is a finite union of closed intervals.*
- (2) *In particular, if the tail bounds the term for all  $k > N$  then  $\mathfrak{C}_{\infty} = \mathfrak{C}_N$  consists of at most  $2^N$  disjoint closed intervals.*
- (3) *If in addition the term exceeds the tail for  $k = 1, \dots, K$ , then  $\mathfrak{C}_{\infty}$  consists of at least  $2^K$  disjoint closed intervals.*

---

<sup>10</sup>A property is said to hold *eventually* for a sequence if there is some place  $K$  in the sequence so that the property holds for *all* later terms—or equivalently, if the property fails to hold for at most a finite number of terms.

As an example, consider the  $p$ -sequence

$$x_k = \frac{1}{k^p}$$

where  $p > 1$  is a fixed exponent. The precise value of the  $n^{\text{th}}$  tail  $X_n$  is hard to determine, but we can take advantage of the standard proof of summability (that is, the integral test) to estimate it and so try to check which terms exceed the associated tails and which tails bound the terms.

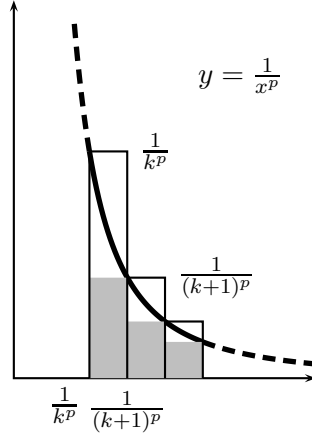


FIGURE 4. Integral Test for  $p$ -series

From Figure 4 we obtain the estimates

$$\int_{n+1}^{\infty} \frac{dx}{x^p} < \sum_{k=n+1}^{\infty} \frac{1}{k^p} < \int_n^{\infty} \frac{dx}{x^p}.$$

Carrying out the integration on either side, we have

$$(5) \quad \frac{(n+1)^{1-p}}{p-1} < X_n < \frac{n^{1-p}}{p-1}.$$

Thus we can guarantee that the  $n^{\text{th}}$  tail exceeds the  $n^{\text{th}}$  term

$$x_n < X_n$$

whenever

$$\frac{1}{n^p} \leq \frac{(n+1)^{1-p}}{p-1}.$$

a condition which can be rewritten

$$p-1 \leq (n+1) \left( \frac{n}{n+1} \right)^p.$$

Fixing  $p > 1$ , the fraction on the right converges to 1, while the first factor goes to infinity, so (for a given exponent  $p$ ) the  $n^{\text{th}}$  tail bounds the  $n^{\text{th}}$  term eventually.

We leave it to the reader to check that the function

$$f_p(x) = (x+1) \left( \frac{x}{x+1} \right)^p$$

is strictly increasing.

However, the condition

$$x_n > X_n$$

is guaranteed to hold whenever

$$\frac{n^{1-p}}{p-1} \leq \frac{1}{n^p}$$

or

$$p-1 \geq n :$$

the  $n^{\text{th}}$  term exceeds the  $n^{\text{th}}$  tail at least for<sup>11</sup>  $n \leq K := \lfloor p \rfloor - 1$ . We then have

**Corollary 7.** *The subsum set of a summable  $p$ -sequence is a finite union of disjoint closed intervals. The number of these intervals is between  $2^K$  and  $2^N$ , where*

- $K$  is the highest integer less than or equal to  $p-1$ , and
- $N$  is the least integer such that

$$p-1 \leq (N+1) \left( \frac{N}{N+1} \right)^p.$$

Proposition 6 takes care of sequences for which the tail eventually bounds the term. The situation is more complicated when the term eventually exceeds the tail, but not immediately.

If at some stage the term exceeds the tail, it is still true that each of the intervals  $J_{\xi_1 \dots \xi_n}$  will split into two subintervals separated by a “gap”. However, if the tail bounded the term at some previous stage, we can no longer assume that the intervals which are splitting are disjoint, so in principle the “gap” created when one of them splits can be covered over by part of another one, so that (at least as far as this part of the set is concerned) no new gap is created in  $\mathfrak{C}_{n+1}$ .

An example of this phenomenon is the sequence

$$\frac{2}{5}, \frac{9}{25}, \frac{12}{125}, \frac{54}{625}, \dots$$

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<sup>11</sup> $\lfloor p \rfloor$  denotes the highest integer  $\leq p$

defined by<sup>12</sup>

$$x_{2k} = \frac{9 \cdot 6^{k-1}}{5^{2k}}$$

$$x_{2k+1} = \frac{2 \cdot 6^k}{5^{2k+1}}.$$

In Figure 5, we have sketched out the sets  $\mathfrak{C}_n$ ,  $n = 1, \dots, 5$  for this sequence. We have pictured the intervals vertically instead of horizontally to take advantage of the page dimensions. The nested rectangles on the left side of the picture represent the various intervals  $J_\xi$ : the passage from a word  $\xi$  of even length to its two subintervals is represented by two rectangles of roughly half the width of the preceding, one lined up with the bottom and left edge and the other lined up with the top and right edge; the projections of these two overlap, so their projection to the right doesn't change the set  $\mathfrak{C}_{2n}$ . The next passage, however, introduces a “gap” into each such subrectangle; we have represented the two new subintervals as rectangles of the same length, lined up horizontally. This time, some gaps appear in the total projection while others do not. A gap formed in going from  $J_\xi$  ( $\xi$  of even length) to its predecessor does not show up in the projection  $\mathfrak{C}_{2n+1}$  if a subinterval inside another  $J_\xi$  covers it up. The pattern of coverups is complicated: in particular, note that the projection of the 64 intervals corresponding to words of length 6 has 23 components: shorter outer ones, and a considerably longer middle one, formed from the projection of 18  $J_\xi$ 's. Nonetheless, the components of  $\mathfrak{C}_n$  do appear to keep breaking up into subintervals, suggesting that at the end there will be infinitely many components to  $\Sigma(\{x_i\}_{i=1}^\infty)$ . In fact, this turns out to be true. To see why, we need to study what happens at the far left of  $\mathfrak{C}_n$  when the term exceeds the tail.

Consider the component of  $\mathfrak{C}_n$  containing 0. Fix an index  $n$ .

Every interval  $J_\xi$  ( $\xi$  a word of length  $n$ ) is a translate to the right of the leftmost interval  $J_{0^n}$  by  $x_\xi$ . Since we have assumed the sequence is non-increasing, the shortest of these translations is given by  $x_n$ . Thus for any  $n$  the interval  $[0, x_n)$  intersects  $J_{0^n}$  but is disjoint from all the other intervals  $J_\xi$  ( $\xi$  a word of length  $n$ ) making up  $\mathfrak{C}_n$ . (See Figure 6)

Suppose now that the  $n^{\text{th}}$  term exceeds the  $n^{\text{th}}$  tail ( $x_n > X_n$ ), so that  $[0, x_{n-1}]$  breaks into two subintervals,  $[0, X_n]$  and  $[x_n, x_n + X_n]$ . The only word  $\xi$  of length  $n-1$  for which  $J_\xi$  intersects  $[0, x_{n-1}]$  is  $\xi = 0^n$ . Thus, the “gap”  $(X_n, x_n)$  introduced into  $J_{0^n}$  when it breaks up into  $J_{0^n 0 = 0^{n+1}}$  and  $J_{0^n 1}$  becomes a gap in  $\mathfrak{C}_n$ . (See lower part of Figure 6.)

A similar argument applies when  $x_\xi$  is the left endpoint of any component of  $\mathfrak{C}_{n-1}$ ; the easiest way to see this is to translate the whole picture using  $x_\xi$  and note that to the left of  $x_\xi$  there is a larger “gap” coming from some earlier separation. Finally, we can use the symmetry of  $\mathfrak{C}_\infty$  under  $x \mapsto X_0 - x$

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<sup>12</sup>We shall see how this mysterious sequence was created in § 5.



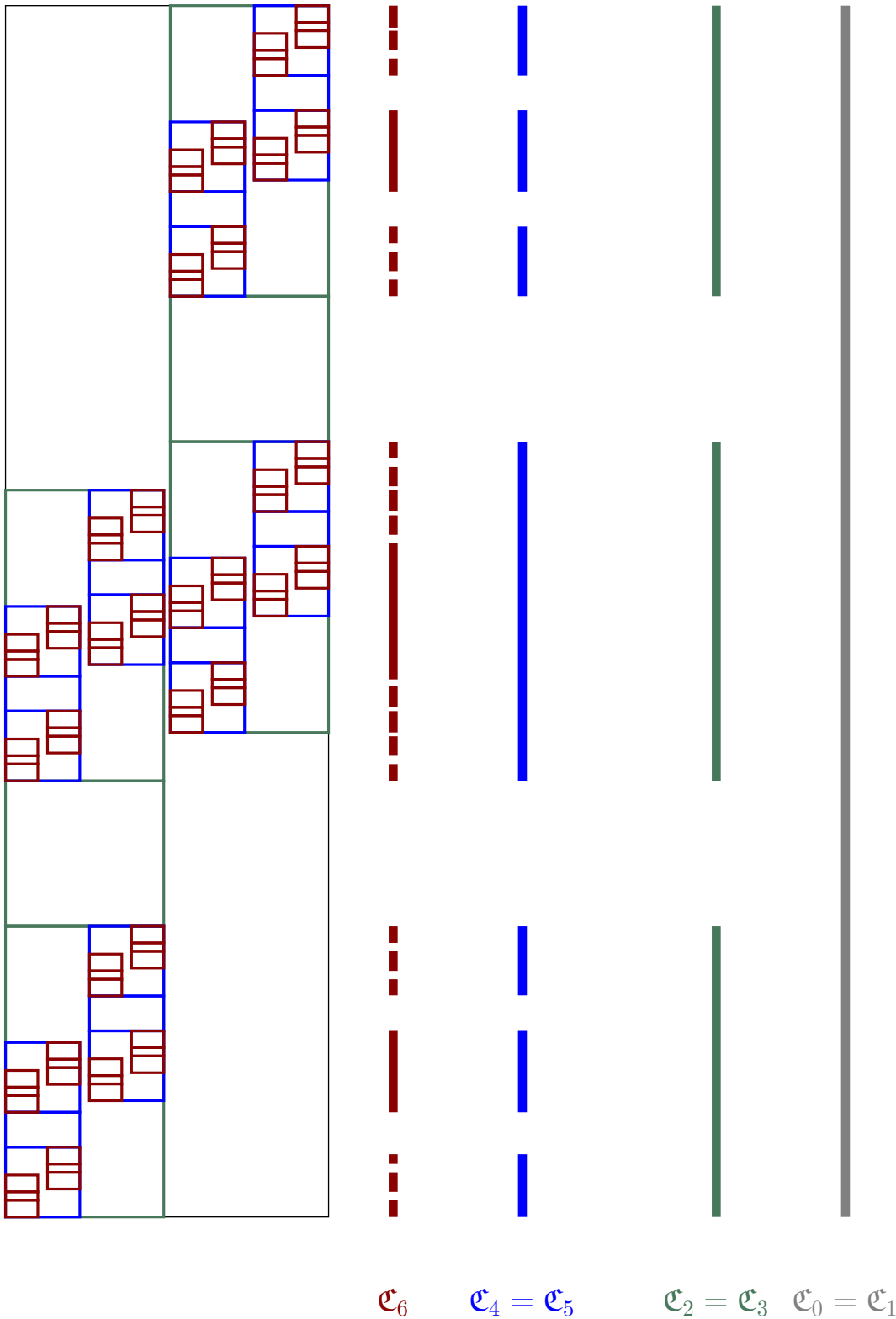


FIGURE 5.  $\mathfrak{C}_n$  for  $\frac{2}{5}, \frac{9}{25}, \frac{12}{125}, \frac{54}{625}, \dots$

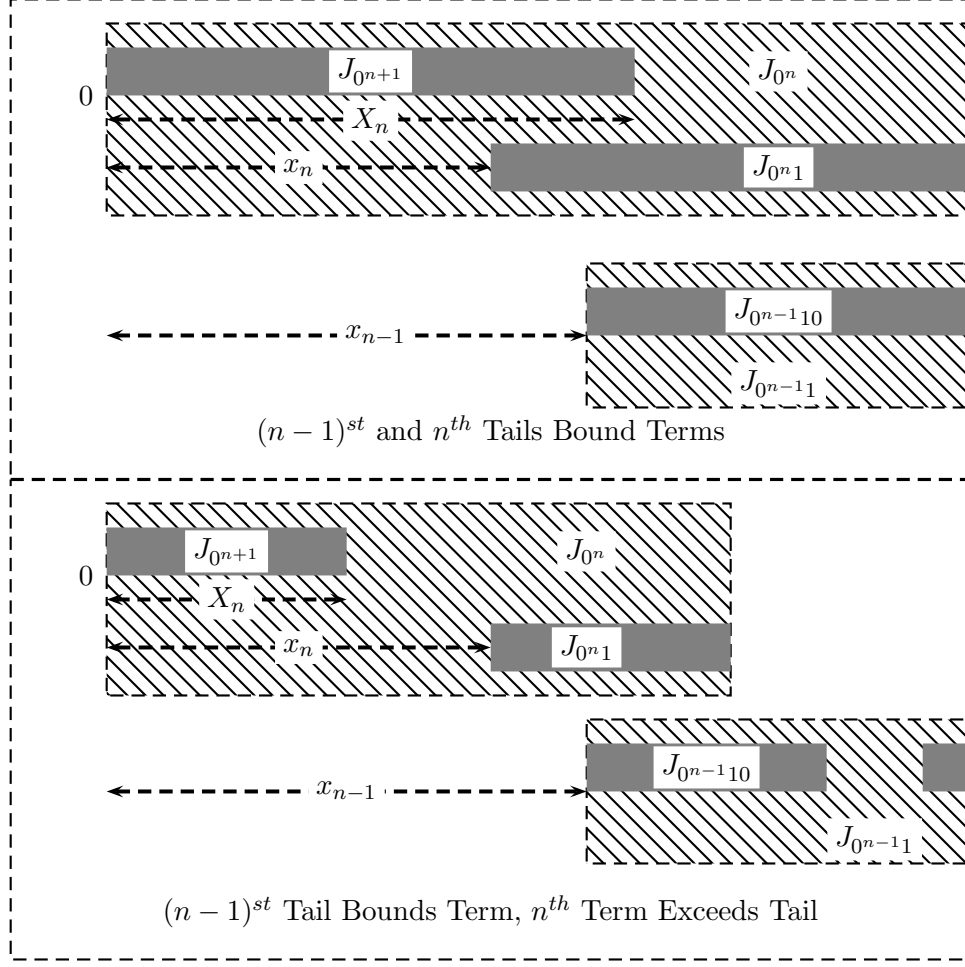


FIGURE 6. The first interval

established in Theorem 3 to draw the same conclusion for the *right* endpoints of components of  $\mathfrak{C}_n$ . This gives us

**Lemma 8.** *Suppose  $\{x_i\}$  is a positive, non-increasing summable sequence.<sup>13</sup> If at stage  $n$  the term exceeds the tail*

$$x_n > X_n$$

*and  $[a, b]$  is a component of  $\mathfrak{C}_{n-1}$ , then  $[a, a + X_n]$  and  $[b - X_n, b]$  are disjoint components of  $\mathfrak{C}_n$ .*

<sup>13</sup>Every positive sequence can be rewritten in non-increasing order without changing the subsum set. However, the sequence of tails—and hence, presumably, the times when the term exceeds the tail—is certainly affected by such a reordering. It is critical for our argument that the sequence be given in non-increasing order before this condition is checked.

**Proposition 9.** *Suppose  $\{x_i\}$  is a positive, non-increasing summable sequence.*

*Then  $\Sigma(\{x_i\}_{i=1}^{\infty})$  has*

- (1) *infinitely many components if the term exceeds the tail infinitely often;*
- (2) *at least  $2^N$  components if the term exceeds the tail  $N$  times.*

Note that (2) generalizes the lower bound given in Proposition 6(3).

In particular,

**Corollary 10.** *The subsum set of a positive, non-increasing summable sequence  $\{x_i\}$  is a finite union of intervals if and only if the tail eventually exceeds the term.*

Proposition 9(1) and Corollary 10 strongly suggest that a subsum set is either a finite union of closed intervals or a Cantor set. However, to show that a subsum set is a Cantor set, we need to show not only that it has infinitely many components, but also that it has empty interior, or equivalently, that every component is a single point. The following observation, which follows from Lemma 8, suggests that this might be true:

**Remark 11.** *Suppose  $\{x_i\}$  is a positive, non-increasing summable sequence. If the term exceeds the tail infinitely often, then each endpoint of every component of each  $\mathfrak{C}_n$  constitutes a one-point component of  $\Sigma(\{x_i\}_{i=1}^{\infty}) = \mathfrak{C}_{\infty}$ .*

Takeya [10] conjectured that the subsum set is a Cantor set if and only if the term exceeds the tail infinitely often; initially, I had the same intuition. However, it turns out that there exist subsum sets with infinitely many components but nonempty interior. We shall study some examples in the next section.

#### 4. CANTORVALS

The following example was analyzed by Guthrie and Nymann in [8] in the process of characterizing the range of an arbitrary finite measure. Consider the positive decreasing summable sequence

$$\frac{3}{4}, \frac{2}{4}, \frac{3}{16}, \frac{2}{16}, \dots$$

that is,

$$x_{2k-1} = \frac{3}{4^k}$$

$$x_{2k} = \frac{2}{4^k}.$$

The tails of this sequence are

$$X_{2k} = \frac{5}{3 \cdot 4^k}, k = 0, 1, \dots$$

$$X_{2k-1} = \frac{11}{3 \cdot 4^k}, k = 1, \dots$$

Since

$$3 < \frac{11}{3}$$

and

$$2 > \frac{5}{3}$$

we see that every even-numbered term exceeds the corresponding tail, so  $\Sigma(\{x_i\}_{i=1}^\infty)$  has infinitely many components, by Proposition 9.

Guthrie and Nymann show that the subsum set contains the interval  $[\frac{3}{4}, 1]$ , as follows. Since the subsum set is closed and every number in  $[\frac{3}{4}, 1]$  can be expanded to base 4 as

$$\frac{3}{4} + \sum_{k=2}^{\infty} \frac{a_k}{4^k}, \quad a_k \in \{0, 1, 2, 3\},$$

it suffices to show that each *finite* sum of this type can be expressed as a subsum of our sequence:

$$x = \frac{3}{4} + \sum_{k=2}^n \frac{a_k}{4^k} = \sum_{k=1}^n \left( \frac{3}{4^k} b_k + \frac{2}{4^k} c_k \right)$$

where  $b_k, c_k \in \{0, 1\}$ . This is easy to do if  $x = \frac{3}{4}$  (*i.e.*,  $n = 1$ ); otherwise, we proceed by induction on  $n$ : assume we can express every number involving only multiples of  $4^k$  with  $k < n$

$$\frac{3}{4} + \sum_{k=2}^{n-1} \frac{a_k}{4^k} \quad (a_k = 0, 1, 2, 3)$$

in this way and write

$$x = x' + \frac{d_k}{4^n}$$

where  $d_k \in \{0, 2, 3\}$  and  $x'$  involves only multiples of  $4^k$  with  $k < n$ . By induction hypothesis,  $x'$  can be written in the desired form:

$$x' = \sum_{k=1}^{n-1} \left( \frac{3}{4^k} b_k + \frac{2}{4^k} c_k \right).$$

Now note that

$$\frac{3}{4^i} + \frac{2}{4^i} = \frac{1}{4^i} + \frac{1}{4^{i-1}}$$

so if  $d_k = 0, 2$ , or  $3$  we are done, while if  $d_k = 1$  then by induction  $x' - \frac{1}{4^{n-1}}$ , which also involves only multiples of  $4^k$  with  $k < n$ , already has such an expression, and hence so does

$$x = \left( x' - \frac{1}{4^{n-1}} \right) + \frac{1}{4^i} + \frac{1}{4^1}.$$

This shows that the subsum set of this example has infinitely many components but nonempty interior.

Before I came across [8], I was sent a different example by Rick Kenyon, namely

$$\frac{6}{4}, \frac{1}{4}, \frac{6}{16}, \frac{1}{16}, \dots$$

or

$$x_{2k-1} = \frac{6}{4^k}$$

$$x_{2k} = \frac{1}{4^k}.$$

We note that this order, while it makes transparent the generating formulas for the sequence, is not monotone: for example,  $x_2 = \frac{1}{4} = \frac{4}{16} < \frac{6}{16} = x_3$ , and  $x_4 = \frac{1}{16} = \frac{2}{32} > x_5 = \frac{6}{64} = \frac{3}{32}$ . For the record, the non-increasing order is

$$\frac{6}{4}, \frac{6}{16}, \frac{1}{4}, \frac{6}{64}, \frac{1}{16}, \frac{6}{256}, \frac{1}{64}, \dots$$

The reader can verify that the term exceeds the tail infinitely often. However, an interesting number-theoretic argument suggested by Kenyon [11, §2] gives a way to generate many examples with nonempty interior and, presumably, infinitely many components (including the Guthrie-Nymann one).

The key observation (in the case of Kenyon's example) is that every congruence class mod 4 can be obtained as a sum of the “digits” 6 and 1, since  $6 \equiv 2 \pmod{4}$  and  $6 + 1 = 7 \equiv 3 \pmod{4}$ . Thus the set of sums of Kenyon's sequence is the set of all reals which can be expressed as “generalized base 4 expansions” using the “digits” 0, 1, 6 and 7:

$$\Sigma(\{x_i\}_{i=1}^\infty) = \left\{ \sum_{i=1}^\infty \frac{a_i}{4^i} \mid a_i \in \{0, 1, 6, 7\} \right\}.$$

**Proposition 12** (R. Kenyon). *Suppose we are given  $n \in \mathbb{N}$  and  $n$  integers  $d_0, d_1, \dots, d_{n-1}$  such that*

$$d_i \equiv j \pmod{n}.$$

*Then the set of “generalized base  $n$  expansions” using these “digits”*

$$\mathcal{S} = \left\{ \sum_{i=1}^\infty \frac{a_i}{n^i} \mid a_i \in \{d_0, \dots, d_{n-1}\} \right\}$$

*has nonempty interior.*

*Proof.* The first step is to confirm the somewhat optimistic intuition that, since the digits include representatives of all the congruence classes mod  $n$ , the finite sums of the form

$$\sum_{i=1}^k \frac{a_i}{n^i}, \quad a_i \in \{d_0, \dots, d_{n-1}\}$$

should, by analogy with the standard case  $d_j = j$ , have fractional parts that include all rational numbers of the form  $\frac{a}{n^k}$ . The “obvious” reasoning we might expect does not apply: for example,  $\frac{1}{4} + \frac{2}{4^2} = \frac{6}{16}$  while  $\frac{1}{4} + \frac{6}{4^2} = \frac{10}{16}$ ;

the difference is not an integer even though  $6 = 2 \pmod{4}$ . However, it *is* true that *different* expressions of this form have *different* fractional parts. To see this, suppose we have two such sums with the same fractional part:

$$\frac{a_1}{n} + \frac{a_2}{n^2} + \cdots + \frac{a_k}{n^k} = \frac{b_1}{n} + \frac{b_2}{n^2} + \cdots + \frac{b_k}{n^k} + N$$

(where each  $a_i$  and  $b_i$  is one of our “digits”  $d_0, \dots, d_{n-1}$ , and  $N \in \mathbb{N}$ ). We can rewrite this as

$$\frac{a_1 - b_1}{n^1} + \frac{a_2 - b_2}{n^2} + \cdots + \frac{a_k - b_k}{n^k} = N$$

and multiply both sides by  $n^k$ :

$$n^{k-1}(a_1 - b_1) + n^{k-2}(a_2 - b_2) + \cdots + n(a_{k-1} - b_{k-1}) + (a_k - b_k) = n^k N.$$

Taking the congruence class of both sides  $\pmod{n}$ , we get

$$a_k - b_k \equiv 0 \pmod{n}.$$

But since the possible digits belong to different congruence classes  $\pmod{n}$ , we must have

$$a_k = b_k.$$

Thus by induction on  $k$ ,  $a_i = b_i$  for  $i = 1, 2, \dots, k$ .

Now, for a given (fixed)  $k$ , there are  $n^k$  sums of the form

$$\sum_{i=1}^k \frac{a_i}{n^i}$$

as well as  $n^k$  fractions of the form  $\frac{a}{n^k}$  with  $0 \leq a < n^k$ . Hence by the pigeonhole principle, congruence  $\pmod{n}$  generates a bijection between the two sets, confirming our intuition.

The second step is then to reinterpret this statement to say that the integer translates of  $\mathcal{S}$  cover the whole real line

$$\bigcup_{k \in \mathbb{Z}} (k + \mathcal{S}) = \mathbb{R}.$$

Finally, we invoke the *Baire Category Theorem*, which in our context says that if a countable union of sets equals  $\mathbb{R}$  then at least one of them has nonempty interior [1].<sup>14</sup> From this we conclude that for at least one integer  $k$ ,  $(k + \mathcal{S})$  has non-empty interior—but since it is a translate of  $\mathcal{S}$ , the same is true of  $\mathcal{S}$ .  $\square$

Having established the existence of subsum sets with infinitely many components but non-empty interior, we should try to understand better the structure of these sets.

<sup>14</sup>This was Baire’s doctoral dissertation; see Dunham’s highly readable account in [6, pp. 184-191]. A more general version of this (involving complete metric spaces), is proved in many basic analysis texts; for example, see [16, Thm. 4.31, pp. 243-5] or [18, Prob. 16, p.40],

Suppose a subsum set  $\Sigma(\{x_i\}_{i=1}^\infty)$  has infinitely many components but non-empty interior. For each  $n$ , we can write  $\Sigma(\{x_i\}_{i=1}^\infty)$  as the union of  $2^n$  translates of the set  $\Sigma_{(n)}$ . Invoking the Baire Category Theorem again (this time in its weaker form, involving a finite union) we conclude that one, and hence all, of these translates has non-empty interior. In particular, each interval  $J_{\xi_1 \dots \xi_n}$  contains a subinterval of  $\Sigma(\{x_i\}_{i=1}^\infty)$ . This means that every point of  $\Sigma(\{x_i\}_{i=1}^\infty)$  is within distance  $X_n$  of some subinterval of  $\Sigma(\{x_i\}_{i=1}^\infty)$ . Since  $X_n \rightarrow 0$ , the subintervals (in particular the non-trivial components) of  $\Sigma(\{x_i\}_{i=1}^\infty)$  are dense. At the same time, Remark 11 tells us that the trivial (*i.e.*, one-point) components of  $\Sigma(\{x_i\}_{i=1}^\infty)$  are also dense, in the sense that every endpoint of a non-trivial component is an accumulation point of trivial components. In addition to Guthrie and Nymann [8], such sets were studied by Mendes and Oliveira [13], in connection with the structure of arithmetic sums of Cantor sets (motivated by the study of bifurcation phenomena in dynamical systems). They dubbed them *Cantorvals*. In their context, three varieties of Cantorvals can arise, but because of the symmetry of subsum sets, the only kind that arises in our context is what they call an *M-Cantorval*. I prefer the more descriptive term *symmetric Cantorval*. Formally:

**Definition 13.** *A **symmetric Cantorval** is a nonempty compact subset  $\mathcal{S}$  of the real line such that*

- (1)  *$\mathcal{S}$  is the closure of its interior (*i.e.*, the nontrivial components are dense)*
- (2) *Both endpoints of any nontrivial component of  $\mathcal{S}$  are accumulation points of trivial (*i.e.*, one-point) components of  $\mathcal{S}$ .*

Our remarks above establish the full topological classification of subsum sets for summable positive sequences, proven by Guthrie and Nymann (with different terminology) in [8]:

**Theorem 14** (Guthrie-Nymann). *The subsum set of a positive summable sequence is one of the following:*

- (1) *a finite union of (disjoint) closed intervals;*
- (2) *a Cantor set;*
- (3) *a symmetric Cantorval.*

Each of the first two categories in Theorem 14 provides a list of possible topological types: in the first case, the number of components determines the topological type, while in the second, all Cantor sets are homeomorphic, by a well-known theorem (see for example [16, pp. 103-4]). It turns out that all (symmetric) Cantorvals are also homeomorphic. This was proved in [8] and stated without explicit proof in [13].

**Proposition 15.** *Any two symmetric Cantorvals are homeomorphic.*

*Proof.* Given two Cantorvals  $\mathfrak{C}$  and  $\mathfrak{C}'$ , first identify the longest component of each; if there is some ambiguity (because several components have the

same maximal length), then pick the leftmost one. There is a unique affine, order-preserving homeomorphism between them.

Note that by definition there are other components of  $\mathfrak{C}$  (*resp.*  $\mathfrak{C}'$ ) on either side of the chosen one. In particular, its complement is contained in two disjoint intervals, one to the right and one to the left, and the part of each Cantorval in each of these intervals is again a Cantorval. Thus, we can apply the same algorithm to pair the longest nontrivial component to the left (*resp.* right) of the chosen one in  $\mathfrak{C}$  with the corresponding one in  $\mathfrak{C}'$ . Continuing in this way, we get an order-preserving correspondence between the non-trivial components of  $\mathfrak{C}$  and those of  $\mathfrak{C}'$ , and an order-preserving homeomorphism between corresponding components. But this means we have an order-preserving continuous mapping from the (dense) interior of  $\mathfrak{C}$  onto the interior of  $\mathfrak{C}'$ . This uniquely extends to a homeomorphism from all of  $\mathfrak{C}$  onto all of  $\mathfrak{C}'$ .  $\square$

Guthrie and Nymann point out that a model symmetric Cantorval can be constructed by following the standard construction of the middle-third Cantor set (removing the middle third of each component at a given stage) but then going back and “filling in” the gaps at *every other* stage.

Figure 7 is a picture of Kenyon’s example, which we have noted is a symmetric Cantorval. It is drawn as a bar graph, generated via a Maple program by Don Plante.

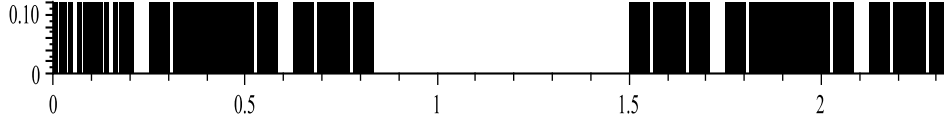


FIGURE 7. Kenyon’s example

## 5. BI-GEOMETRIC SEQUENCES

We saw in § 3 that the subsum set of a geometric sequence is either an interval or a Cantor set, because the relation between the term and the tail is always the same. We can construct examples which exhibit any particular pattern of alternation between the two possible relations by looking at the sequence of sets  $\mathfrak{C}_n$  in a different way, in terms of ratios.

To be precise, given a sequence  $\{x_i\}$  of terms, let us look at the associated sequence of tails,  $\{X_i\}$ , and for each index  $i$ , consider the proportion of  $X_i$  taken up by  $x_{i+1}$ :

$$\rho_i := \frac{x_{i+1}}{X_i}$$



or equivalently

$$(6) \quad x_{i+1} = \rho_i \cdot X_i$$

Then, since

$$X_i = x_{i+1} + X_{i+1},$$

we have

$$(7) \quad X_{i+1} = (1 - \rho_i) \cdot X_i.$$

Conversely, the sequence of ratios  $\{\rho_i\}$  together with the total sum  $X_0$  determines the sequence  $\{x_i\}$  recursively, via the initial condition

$$x_1 = \rho_0 X_0$$

and the relation

$$x_{i+1} = \rho_i (\rho_{i-1}^{-1} - 1) x_i.$$

Equivalently,  $x_i$  can be given by an explicit formula:

$$(8) \quad x_i = \rho_i \prod_{j=0, \dots, i-1} (1 - \rho_j) X_0.$$

The initial (total) sum  $X_0$  is simply a scaling factor, so to determine what kind of set occurs we can *assume that the total sum is  $X_0 = 1$* .

Now, at each stage, the term and tail are determined from the previous tail by (6) and (7), from which it is easy to see that

- the sequence  $\{x_i\}$  is non-increasing if and only if for every  $i$

$$(9) \quad \rho_i \leq \frac{\rho_{i-1}}{1 - \rho_{i-1}};$$

- the  $n^{th}$  term exceeds the  $n^{th}$  tail ( $x_n > X_n$ ) if and only if

$$\rho_{n-1} < \frac{1}{2}$$

and (equivalently)

- the  $n^{th}$  tail bounds the  $n^{th}$  term ( $x_n \leq X_n$ ) if and only if

$$\rho_{n-1} \geq \frac{1}{2}.$$

So one way to create a sequence for which both possibilities occur infinitely often is to pick two ratios,

$$(10) \quad 0 < \alpha < \frac{1}{2} < \beta < 1,$$

and to set

$$\rho_i = \begin{cases} \alpha & \text{for even } i, \\ \beta & \text{for odd } i. \end{cases}$$

This leads to the sequence

$$\begin{aligned} x_{2k} &= \beta(1 - \alpha)^k(1 - \beta)^{k-1} \\ x_{2k+1} &= \alpha(1 - \alpha)^k(1 - \beta)^k. \end{aligned}$$

A sequence defined in this way spiritually resembles a geometric sequence, except that it involves two distinct ratios, so we might refer to it as a **bi-geometric sequence**. An obvious generalization of this idea, which could be called a *multi-geometric sequence*, is one where the sequence of ratios  $\rho_i$  is periodic; we could refer to a sequence for which  $\rho_{i+m} = \rho_i$  for some fixed  $m > 0$  and all  $i$  as an *m-geometric sequence*. We shall deal only with bi-geometric sequences in this paper.

We have seen three examples of bi-geometric sequences earlier in this paper. The sequence whose sets  $\mathfrak{C}_n$  are shown in Figure 5

$$\begin{aligned} x_{2k} &= \frac{9 \cdot 6^{k-1}}{5^{2k}} \\ x_{2k+1} &= \frac{2 \cdot 6^k}{5^{2k+1}}. \end{aligned}$$

was constructed so that

$$\alpha = \frac{2}{5}, \quad \beta = \frac{3}{5}$$

while both the Guthrie-Nymann and Kenyon examples have

$$\alpha = \frac{9}{20}, \quad \beta = \frac{6}{11}.$$

The first observation above says that, in order to have a non-increasing sequence  $\{x_i\}$ , we also need  $\alpha$  and  $\beta$  to satisfy

$$(11) \quad \alpha \leq \frac{\beta}{1 - \beta}$$

$$(12) \quad \beta \leq \frac{\alpha}{1 - \alpha}.$$

Note that, since we require  $\beta > \frac{1}{2}$ , Equation (12) puts further limitations on the possible values of  $\alpha$ :

$$\frac{1}{2} < \beta \leq \frac{\alpha}{1 - \alpha}$$

forces

$$1 - \alpha < 2\alpha$$

or

$$(13) \quad \alpha > \frac{1}{3}.$$

By contrast, Equation (11) puts no further restrictions on  $\beta$ .

We can try to analyze the subsum set of a bi-geometric sequence by using the idea of an *iterated function system* ([3], [7]). Suppose we have a sequence

defined in terms of two parameters  $\frac{1}{3} < \alpha < \frac{1}{2} < \beta < 1$ , subject to (11) and (12), by

$$\begin{aligned} x_{2k} &= \beta(1 - \alpha)^k(1 - \beta)^{k-1} \\ x_{2k+1} &= \alpha(1 - \alpha)^k(1 - \beta)^k. \end{aligned}$$

The sets  $\mathfrak{C}_0$  and  $\mathfrak{C}_1$  are the same, and we can describe the set  $\mathfrak{C}_2$  in terms of the set  $\mathfrak{C}_0 = [0, 1]$  as the union of four intervals  $J_{ij}$ ,  $i, j \in \{0, 1\}$ , each of length  $X_2 = (1 - \alpha)(1 - \beta)$ , with respective endpoints

$$\begin{aligned} x_{00} &= 0 \\ x_{10} &= x_1 = \alpha \\ x_{01} &= x_2 = \beta(1 - \alpha) \\ x_{11} &= x_1 + x_2 = \alpha + \beta - \alpha\beta. \end{aligned}$$

Each of these intervals can be obtained from the basic interval  $\mathfrak{C}_0 = [0, 1]$  by scaling and translation; specifically, we can define four affine functions, all with the same scaling factor

$$\lambda = (1 - \alpha)(1 - \beta) :$$

$$\begin{aligned} \varphi_{00}(x) &= \lambda x \\ \varphi_{01}(x) &= x_2 + \lambda x \\ &= \beta(1 - \alpha) + \lambda x \\ \varphi_{10}(x) &= x_{10} + \lambda x \\ &= \alpha + \lambda x \\ \varphi_{11}(x) &= x_1 + x_2 + \lambda x \\ &= \alpha + (1 - \alpha)\beta + \lambda x. \end{aligned}$$

Then it is easy to see that, in terms of our earlier notation,

$$J_{\xi_i \xi_j} = \varphi_{ij}(\mathfrak{C}_0).$$

But our recursive relations for  $x_k$  and  $X_k$  repeat every two steps, and hence we get recursive definitions of the sets  $\mathfrak{C}_k$  and  $J_\xi$ :

$$\mathfrak{C}_{2k} = \mathfrak{C}_{2k-1} = \bigcup_{i,j=0}^1 \varphi_{ij}(\mathfrak{C}_{2k-1} = \mathfrak{C}_{2k-2});$$

more specifically, for each word  $\xi$  of length  $2k$  in zeroes and ones, if its initial  $2k - 2$ -word is  $\tilde{\xi}$  and last two entries are  $i, j \in \{0, 1\}$ , then

$$J_{\xi=\tilde{\xi}ij} = \varphi_{ij}(\tilde{\xi}).$$

The various overlaps between images of  $\varphi_{01}$  and  $\varphi_{10}$  make it difficult to carry out a careful analysis of the sets  $\mathfrak{C}_n$  in general. However, one easy

observation allows us to conclude in certain cases that the set  $\mathfrak{C}_\infty$  is a Cantor set. At each stage, the set-mapping

$$\mathfrak{C}_{2k} \mapsto \mathfrak{C}_{2k+2} = \varphi_{00}(\mathfrak{C}_{2k}) \cup \varphi_{01}(\mathfrak{C}_{2k}) \cup \varphi_{10}(\mathfrak{C}_{2k}) \cup \varphi_{11}(\mathfrak{C}_{2k})$$

first scales  $\mathfrak{C}_{2k}$  by the factor  $\lambda = (1 - \alpha)(1 - \beta)$ , duplicates four copies of the scaled version, then lays them down (with some overlap). Ignoring the overlap, we can assert that the total of the lengths of the intervals making up  $\mathfrak{C}_{2k+2}$  is less than  $4\lambda$  times the corresponding measure for  $\mathfrak{C}_{2k}$ . In particular, the longest interval in  $\mathfrak{C}_{2k}$  will have length at most  $(4\lambda)^k$ . This allows us to formulate

**Remark 16.** *Suppose  $\{x_i\}$  is a bi-geometric sequence with ratios*

$$0 < \alpha < \frac{1}{2} < \beta < 1$$

*satisfying Equation (12).*

*If*

$$(14) \quad \lambda := (1 - \alpha)(1 - \beta) < \frac{1}{4},$$

*then  $\Sigma(\{x_i\}_{i=1}^\infty)$  is a Cantor set.*

This shows in particular that our first example, shown in Figure 5, yields a Cantor set, since  $\lambda = \frac{6}{25} < \frac{1}{4}$ . By contrast, the Guthrie-Nymann and Kenyon examples both have  $\lambda = \frac{11}{20} \cdot \frac{5}{11} = \frac{1}{4}$ .

In Figure 8 we have sketched the parameter space  $(\alpha, \beta) \in [0, 1] \times [0, 1]$  for bi-geometric sequences. Our discussion above concerned the upper-left quarter of this square,  $[0, \frac{1}{2}] \times [\frac{1}{2}, 1]$ , characterized by the inequalities (10), but by interchanging the roles of  $\alpha$  and  $\beta$  were necessary we can extend it to the whole square. The hatched areas are excluded by the requirement that the sequence  $\{x_i\}$  be non-decreasing (Equation (11) and (12)). The upper gray area is where Equation (14) holds, guaranteeing that  $\Sigma(\{x_i\}_{i=1}^\infty)$  is a Cantor set. Note that the two examples of Cantorvals (Guthrie-Nymann and Kenyon) both correspond to a point on the boundary of this region, where  $\lambda = \frac{1}{4}$ .

The lower gray area is where both  $\alpha$  and  $\beta$  are at most equal to  $\frac{1}{2}$ , which means the tail always bounds the term—so  $\mathfrak{C}_n = [0, 1]$  for all  $n$ . This leaves the two white regions (labeled with a question marks) where one ratio is at most  $\frac{1}{2}$  while the other is greater than  $\frac{1}{2}$ , where our analysis so far cannot completely determine the topology of the subsum set; however, we do know that in this region the subsum set has infinitely many components, so for each bi-geometric sequence coming from parameters in this interval, the subsum set is either a Cantor set or a symmetric Cantorval. However we have not developed a test to distinguish, in general, which possibility a particular example exhibits. In fact, I don't know if there are Cantorval examples with  $\lambda > \frac{1}{4}$ , or, in the other direction, if there are any bi-geometric sequences with parameters in the white region which yield Cantor sets.

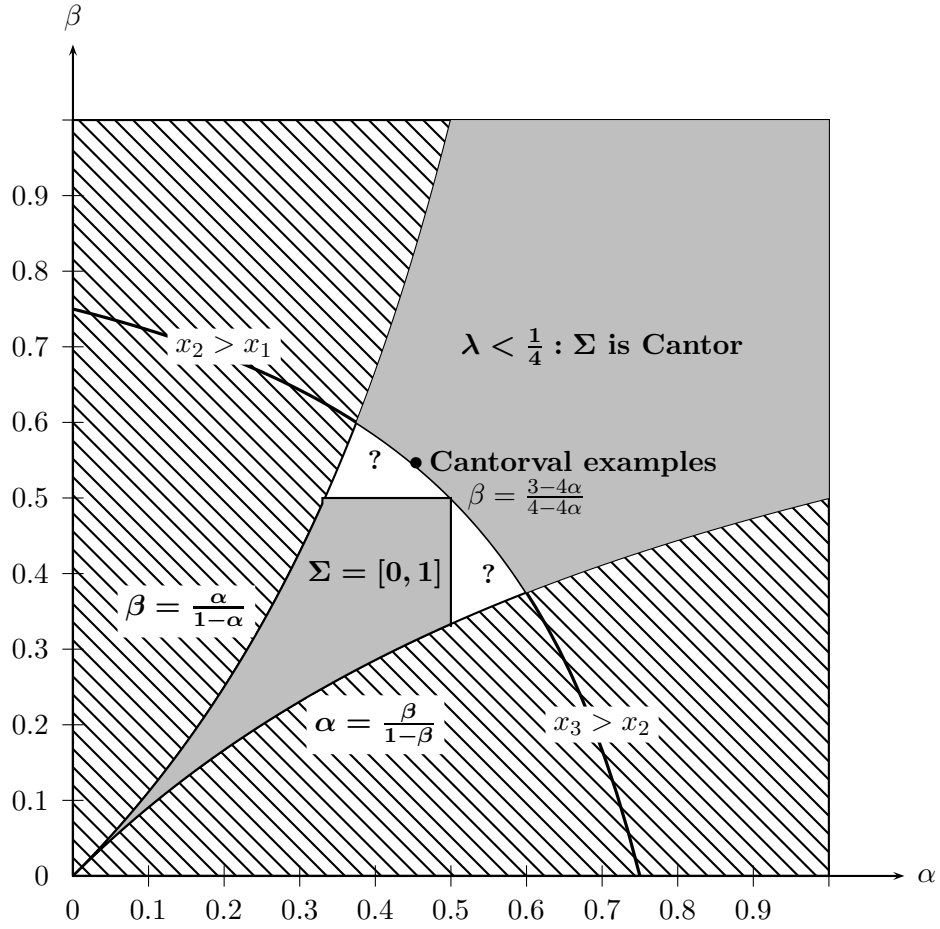


FIGURE 8. Bi-geometric sequences

## 6. SEQUENCES WITH VARYING SIGN

We turn now to the general case, when some terms are positive while others are negative. Here we take advantage of another observation, given by Riemann in [17] but attributed by him to Dirichlet: let us separate out the positive terms of  $x_i$  as  $\{x_i^+\}$  and the negative terms as  $\{x_i^-\}$ . Since the terms of each of these two sums have constant sign, we can define

$$\begin{aligned} \sum x_i^+ &= X^+ \in [0, \infty] \\ \sum x_i^- &= X^- \in [-\infty, 0]. \end{aligned}$$

We can distinguish three possible configurations:

- If both  $X^-$  and  $X^+$  are finite, the sequence is **absolutely summable** ( $\sum_{i=0}^{\infty} |x_i|$  converges), because

$$\sum_{i=0}^{\infty} |x_i| \leq |X^-| + X^+.$$

Recall that as a consequence every reordering of the sequence sums to the same (finite) number.

- If both  $X^-$  and  $X^+$  are infinite, the sequence is **conditionally summable**. It is a standard fact (attributed to Riemann) that if a series converges while the corresponding series of absolute values diverges, then by rearranging the order of the terms we can get a series summing to any real number, or diverging to either  $+\infty$  or  $-\infty$ . Riemann's informal proof of this fact [17, §3] rests on the observation that in this case both  $X^-$  and  $X^+$  are infinite.
- If one is finite and the other infinite, we will call the sequence **unconditionally unsummable**. In this case, *every* reordering gives rise to a divergent series; for example, if  $X^+ = \infty$  and  $X^-$  is finite, then a partial sum of positive terms can be made arbitrarily large, while including negative terms as well can at worst lower this sum by  $|X^-|$ , so any rearrangement diverges to  $\infty$ .

In the absolutely summable case, Kakeya [10] stated without proof that  $\Sigma(\{x_i\}_{i=1}^{\infty})$  equals the interval  $[X^-, X^+]$  if and only if all the tails bound the sums for the sequence of absolute values  $\{|x_k|\}$ . Hornich [9] took this further: again assuming that the sequence is absolutely summable (so both  $X^{\pm}$  are finite, and given a subsequence  $\{y_i\}$  of our sequence, consider the translated sum of its absolute values

$$\sum |y_i| + X^- = \sum |y_i^+| + \sum |y_i^-| + X^- = \sum y_i^+ + \sum |y_i^-| - \sum_k |x_k^-|$$

where the last summand is the sum of the absolute values of *all* the *negative* terms of the original sequence. If we combine the last two sums, the terms  $y_i$  in the subsequence get cancelled, leaving the sum of all the negative terms which are *excluded* from the subsequence. This of course is another subsum of our sequence. Furthermore, *every* subsum of the full sequence can be expressed in this way, which shows that the subsum set of the (absolutely summable) sequence  $\{x_i\}$  is the translate by  $X^-$  of the subsum set of the sequence  $\{|x_i|\}$  of absolute values.

**Proposition 17** (Hornich). *If  $\{x_i\}$  is an absolutely summable sequence, then*

$$\Sigma(\{x_i\}_{i=1}^{\infty}) = \Sigma(\{|x_k|\}_{k=1}^{\infty}) + X^-.$$

This means that the criteria we gave in Theorem 4, Proposition 6 and Corollary 10 can be applied to the (positive) sequence of absolute values to determine the topology of the subsum set of the original, variable sign but absolutely summable sequence.

Finally, if our original sequence is not absolutely summable, we can easily specify the subsum set. In this case we know that at least one of  $X^\pm$  is infinite. We concentrate on the case  $X^+$  infinite; the other case is analogous. Since the subsequence of positive terms is not summable, by Theorem 1  $\Sigma(\{x_i^+\}) = [0, \infty)$ : we can obtain any positive number as the sum of a subsequence of *positive* terms. If  $X^-$  is finite, we can obtain any number in  $[X^-, \infty)$  by adding a positive number to  $X^-$ ; if it is infinite, we can obtain any *negative* number as the sum of some subsequence of *negative* terms—so  $\Sigma(\{x_i\}_{i=1}^\infty) = \mathbb{R}$  in this case.

With a little abuse of notation and sneaky reinterpretation, we can formulate a general characterization of all subsum sets.

The abuse of notation is that we will allow closed interval notation with one or both endpoints infinite; it will be understood that in such a case the square bracket at that end should be replaced by a round parenthesis.

The sneaky reinterpretation is simply this: if a positive sequence is not summable, then *every* “tail” is infinite, so bounds any term.

With these tweaks, we can state a general result, extending Theorem 14:

**Theorem 18.** *Given a null sequence  $x_k \rightarrow 0$ , let  $X^+$  (resp.  $X^-$ ) be the (possibly infinite) sum of all the positive (resp. negative) terms. Then the subsum set  $\Sigma(\{x_i\}_{i=1}^\infty)$  is a closed, perfect set whose convex hull is the interval  $[X^-, X^+]$ , and which is symmetric with respect to reflection across the midpoint of this interval.*

Furthermore, denote the sequence of absolute values of our terms by

$$a_k = |x_k|, \quad k = 1, 2, \dots$$

and its tails by

$$A_k = \sum_{i>k} a_i.$$

Then:

- (1) *If the tail bounds the term*

$$a_k \leq A_k$$

*for all  $k > K$ , and the number of terms which exceed the tail*

$$a_k > A_k$$

*is  $N$ , then  $\Sigma(\{x_i\}_{i=1}^\infty)$  is the union of between  $2^N$  and  $2^K$  disjoint closed intervals.*

- (2) *If the term exceeds the tail infinitely often, then  $\Sigma(\{x_i\}_{i=1}^\infty)$  is either a Cantor set or a symmetric Cantorval. In particular, if the term always exceeds the tail, then  $\Sigma(\{x_i\}_{i=1}^\infty)$  is a Cantor set.*

## 7. OPEN QUESTIONS

We have found necessary and sufficient conditions for  $\Sigma(\{x_i\}_{i=1}^\infty)$  to be a finite union of intervals. By contrast, in the fractal case, we have some sufficient conditions for  $\Sigma(\{x_i\}_{i=1}^\infty)$  to be a Cantor set, but we have not found

any general criterion that distinguishes Cantor sets from Cantorvals. Furthermore, in the fractal case, it would be interesting to get some information about the Lebesgue measure or dimension of  $\Sigma(\{x_i\}_{i=1}^\infty)$ . Some results in this direction are indicated by Hornich [9].

One might also be tempted to ask about the analogous question for null sequences in the complex plane (or more generally points in  $\mathbb{R}^n$ ). In this context, the analogues of the sets  $\mathfrak{C}_n$  will be unions of rectangles (if we think coordinatewise) or discs (if we think of distance), and the analysis of translations will be made more complicated by the need to consider directions as well as distances. Who knows where that might lead?

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